The Interaction Lagrangian for two spin $1/2$ elementary Dirac particles

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ABSTRACT

It has been recently shown* that the spacetime symmetry group of a Dirac particle is larger than the Poincaré group. It also contains spacetime dilations and local rotations. In the quantum representation it becomes $\mathcal{W}\otimes SU(2)$, where $\mathcal{W}$ is the Weyl group (Poincaré group including spacetime dilations) and $SU(2)$ is the unitary representation of the local rotation group. In this work we obtain the interaction Lagrangian for two Dirac particles, which is invariant under this enlarged spacetime group. We analyze the interaction between two Dirac particles, and show that it is possible the existence of metastable bound states for particles of the same charge, provided some initial conditions are fulfilled.

*The space-time symmetry group of a spin 1/2 elementary particle *J. Phys. A: Math. and Gen.* **39**, 4291 (2006)

Atomistic hypothesis
Matter cannot be divided indefinitely. After a finite number of steps we reach a final and indivisible object. We call it an **elementary particle**.

Definition: An **elementary particle** is a mechanical system without excited states. We can destroy it but we can never modify its structure. All its possible states are only kinematical modifications of any one of them.

If the state of an elementary particle changes, it is always possible to find another inertial observer who describes the particle in the same state as in the previous instant.

Corollary

The kinematical space of an elementary particle is necessarily a homogeneous space of the kinematical group associated to the restricted Relativity Principle.

The kinematical variables are $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$, which are interpreted as the time, position of the charge, velocity of the charge and orientation of the system. The system which satisfies Dirac equation is such that $u = c$.

These 9 variables are the non compact variables $t, \boldsymbol{r}$ and the dimensionless compact variables $\widetilde{\theta}, \widetilde{\phi}$ which represent the direction of the velocity $\boldsymbol{u}$ and the α, θ, ϕ which represent a normal parameterization of the orientation of the body frame $\boldsymbol{e}_i$.

Dirac equation corresponds to the classical relationship

$$H - \boldsymbol{u} \cdot \boldsymbol{P} - \boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right) = 0.$$

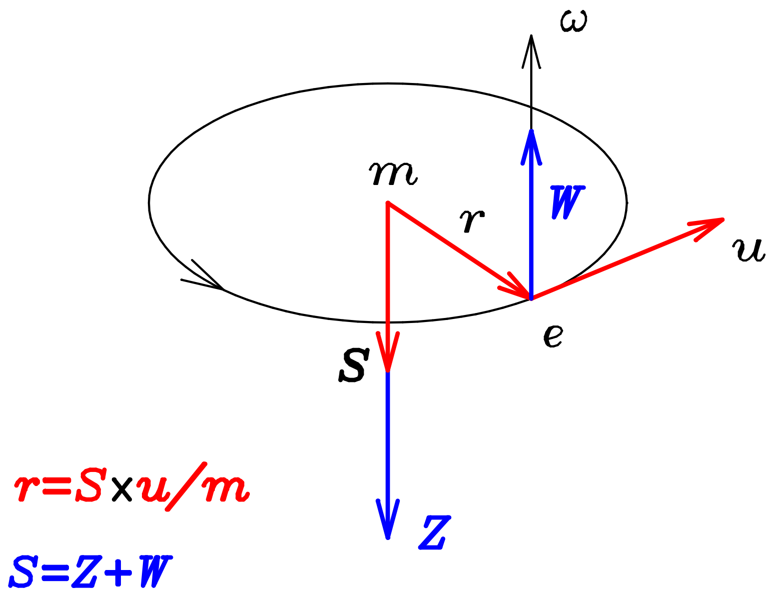
where the spin has a twofold structure

$$\boldsymbol{S} = \boldsymbol{u} \times \frac{\partial L}{\partial \dot{\boldsymbol{u}}} + \frac{\partial L}{\partial \boldsymbol{\omega}} = \boldsymbol{Z} + \boldsymbol{W}.$$

- $\boldsymbol{Z}$ is the Zitterbewegung part of the spin

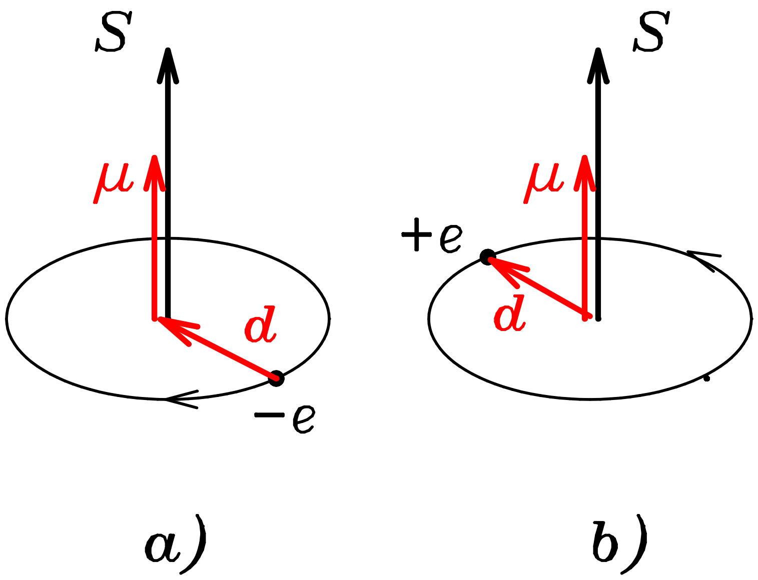
- $\boldsymbol{W}$ is the Rotational part of the spin

A Dirac particle



Total spin $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, is 1/2 and the zitterbewegung part $\boldsymbol{Z}$ quantizes with $z = 0, 1$, while $\boldsymbol{W}$ contributes with $w = 1/2$.

Chirality



Particle and antiparticle have the same mass and spin and also the same electric and magnetic dipole with the same relative orientation with respect to the spin.

Matter is lefthanded and antimatter is righthanded.

The enlarged spacetime symmetry group

If the particle has mass m and spin S we can define a length scale $R = S/mc$ and a time scale $T = S/mc^2$, so that all kinematical variables can be taken dimensionless.
The orientation is also arbitrary and the enlarged spacetime symmetry group is

$$\mathcal{W} \otimes SO(3)_L$$

where $\mathcal{W}$ is the Weyl group (Poincaré group and spacetime dilations) and $SO(3)_L$ is the local rotation group.
The Dirac particle is invariant under spacetime dilations and local rotations

The Interaction Lagrangian

The kinematical space of two Dirac particles is spanned by

$$\{t_a, \boldsymbol{r}_a, \beta_a, \boldsymbol{u}_a, \boldsymbol{\alpha}_a\}, \qquad a = 1, 2.$$

We assume that the Lagrangian which describes the compound system is of the form $L = L_1 + L_2 + L_I$.

Assumption: The spin of the elementary particles cannot be modified

Because the two particles are elementary and we assume the interaction does not modify its internal structure, the interaction Lagrangian L_I cannot be a function of $\dot{\boldsymbol{u}}_a$ and of $\dot{\boldsymbol{\alpha}}_a$ or equivalently $\boldsymbol{\omega}_a$. If it is going to be invariant under the local $SU(2)$ group of local rotations, then it has to be also independent of $\boldsymbol{\alpha}_a$. Otherwise the spin definition of each particle will be modified. It thus remains the same as in the free case

$$\boldsymbol{S}_a = \boldsymbol{u}_a \times \frac{\partial L_a}{\partial \dot{\boldsymbol{u}}_a} + \frac{\partial L_a}{\partial \boldsymbol{\omega}_a} = \boldsymbol{Z}_a + \boldsymbol{W}_a, \quad a = 1, 2.$$

The interaction Lagrangian will thus be a function of

$$L_I = L_I(t_a, \boldsymbol{r}_a, \dot{t}_a, \dot{\boldsymbol{r}}_a),$$

and a homogeneous function of first degree of the derivatives $\dot{t}_a, \dot{\boldsymbol{r}}_a$, $a = 1, 2$. If it is going to be invariant under $\mathcal{W} \otimes SU(2)$, if we call $x_a^\mu \equiv (t_a, \boldsymbol{r}_a)$, then we get the two terms are Poincaré invariant

$$\eta_{\mu\nu}\dot{x}_1^\mu \dot{x}_2^\nu, \quad \eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu).$$

$$L_I = g\sqrt{\frac{\eta_{\mu\nu}\dot{x}_1^\mu \dot{x}_2^\nu}{\eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu)}} = g\sqrt{\frac{\dot{t}_1\dot{t}_2 - \dot{\boldsymbol{r}}_1\cdot\dot{\boldsymbol{r}}_2}{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2 - (t_2 - t_1)^2}}$$

where g is a coupling constant with dimensions of action. Incidentaly we can also see that the Lagrangian is also invariant under the interchange $1 \leftrightarrow 2$.

Synchronous time description

Once an inertial observer is fixed it can make a synchronous time description, i.e. to use as evolution parameter the own observer's time t which is the same as the two time variables t_1 and t_2. In this case

$$L_I = g\sqrt{\frac{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}} = g\frac{\sqrt{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}}{r}$$

where $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the instantaneous separation between the corresponding charges in this frame.

An average over the charge position and velocity in the center of mass of one of the particles imply that the interaction becomes the instantaneous Coulomb interaction, between the center of mass of the first particle (which is also the average position of its charge) and the charge position of the other. The average over the other then corresponds to the interaction of two spinless point particles when the spin structure is neglected.

It is suggesting that $g \sim \pm e^2$ in terms of the electric charge of each particle.

Analysis of a 2-particle system

The dynamical equation of a free Dirac particle is a fourth-order differential equation for the position of the charge which can be separated into a system of coupled second order differential equations for the center of mass $\boldsymbol{q}$ and center of charge $\boldsymbol{r}$ in the form:

$$\ddot{\boldsymbol{q}} = 0, \quad \ddot{\boldsymbol{r}} = \frac{1 - \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{r}}}{(\boldsymbol{q} - \boldsymbol{r})^2}(\boldsymbol{q} - \boldsymbol{r})$$

In the case of interaction the second equation remains the same because it corresponds to the definition of the center of mass position which is unchanged by the interaction. The first equation for particle a is going to be replaced by $d\boldsymbol{p}_a/dt = \boldsymbol{F}_a$ where $\boldsymbol{p}_a$ is the corresponding linear momentum of each particle expressed as usual in terms of the center of mass velocity

$$\boldsymbol{p}_a = \gamma(\dot{\boldsymbol{q}}_a) m \dot{\boldsymbol{q}}_a, \qquad \gamma(\dot{\boldsymbol{q}}_a) = (1 - \dot{\boldsymbol{q}}_a^2)^{-1/2},$$

and the force $\boldsymbol{F}_a$ is computed from the interaction Lagrangian

$$\boldsymbol{F}_a = \frac{\partial L_I}{\partial \boldsymbol{r}_a} - \frac{d}{dt}\left(\frac{\partial L_I}{\partial \boldsymbol{u}_a}\right)$$

For particle 1 takes the form:

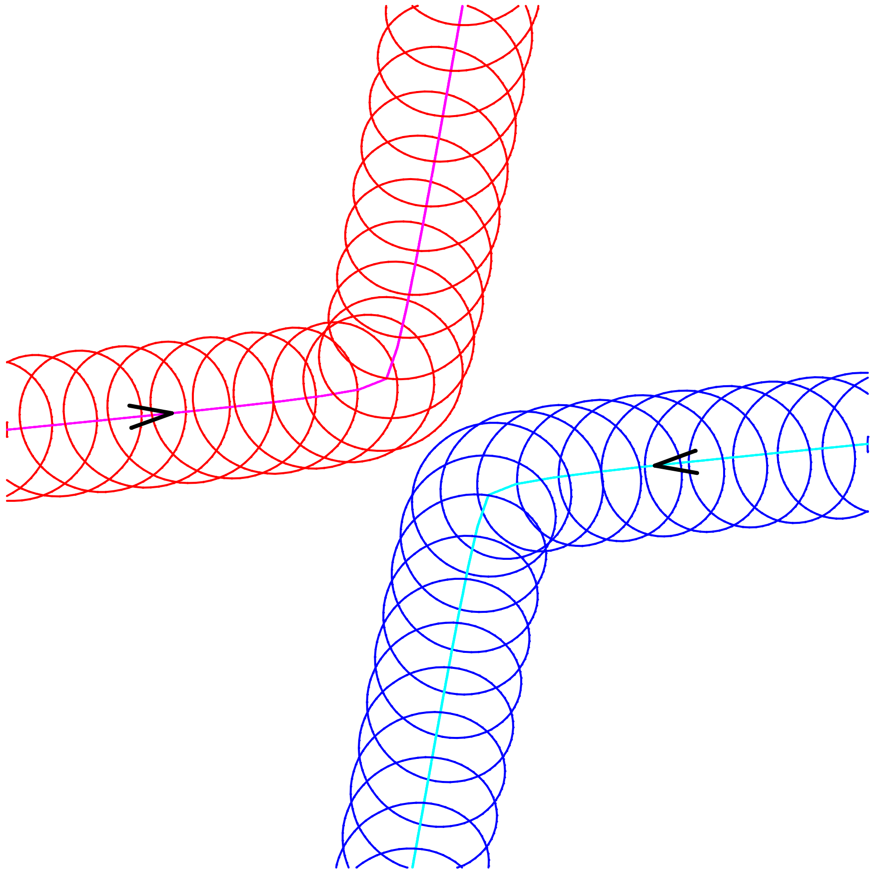
$$\boldsymbol{F}_1 = -g\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\sqrt{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2} + \frac{d}{dt}\left(\frac{g\boldsymbol{u}_2}{2|\boldsymbol{r}_1 - \boldsymbol{r}_2|\sqrt{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}}\right)$$

there are velocity terms which behave like $1/r^2$ and acceleration terms which go as $1/r$. Then the system of second order differential equations to be solved are

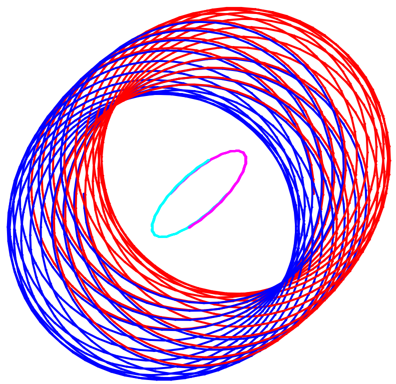
$$\ddot{\boldsymbol{q}}_a = \frac{\alpha}{\gamma(\dot{\boldsymbol{q}}_a)}\left(\boldsymbol{F}_a - \dot{\boldsymbol{q}}_a(\boldsymbol{F}_a \cdot \dot{\boldsymbol{q}}_a)\right) \qquad (1)$$

$$\ddot{\boldsymbol{r}}_a = \frac{1 - \dot{\boldsymbol{q}}_a \cdot \dot{\boldsymbol{r}}_a}{(\boldsymbol{q}_a - \boldsymbol{r}_a)^2}(\boldsymbol{q}_a - \boldsymbol{r}_a), \quad a = 1, 2 \qquad (2)$$

where α is the fine structure constant once all the variables are dimensionless.



The trajectories of the centers of mass and charge of two spinning particles with an initial center of mass velocity $v = 0.1$ and a small impact parameter.



Bound motion of the centers of mass and charge of two spinning particles with parallel spins and with a center of mass velocity $v \leq 0.01$, for an initial separation between the centers of masses $0.2\times$Compton's wavelength.

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